TMA4170 Fourier Analysis

Weyl's ergodic theorem

Lemma (Weyl):

$\gamma \in (0,1) \setminus \mathbb{Q}$, $x \in \mathbb{R}$, $f \in C(\mathbb{R})$ 1-periodic $\Longrightarrow$ $\displaystyle\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x+n\gamma) = \int_0^1 f(y)\,dy$

Equivalently:

Dynamical system $x_n = \langle x + n\gamma \rangle$, $n = 0, 1, \dots$ is ergodic.

$$a = \underset{\substack{\text{\rotatebox{90}{$\in$}}\\ \mathbb{Z}}}{[a]} + \underset{\substack{\text{\rotatebox{90}{$\in$}}\\ [0,1)}}{\langle a \rangle}, \quad \text{integer + frac. part}$$

Corollary:

$\gamma \in (0,1) \setminus \mathbb{Q}$, $(a,b) \subset (0,1)$, $x \in [0,1)$ $\Longrightarrow$ $\displaystyle\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{(a,b)}(x+n\gamma) = b-a$

1-periodic extension

Weyl's equidistribution theorem

$\{\xi_n\}_{n=1}^{\infty} \subset [0,1)$ equidistributed if $\forall\, (a,b) \subset [0,1)$,

$$\underbrace{\lim_{N\to\infty} \frac{\#\{1 \le n \le N : \xi_n \in (a,b)\}}{N}}_{\text{frequency of times } \xi_n \in (a,b) \text{ for } n \le N} = \underbrace{b-a}_{\frac{|(a,b)|}{|[0,1)|}}$$

$\#A$ = no. of points in A

Theorem (Weyl):

$$\boxed{\gamma \in (0,1)\setminus \mathbb{Q}\,,\quad x \in [0,1) \implies \{\langle x + n\gamma \rangle\}_{n=1}^{\infty} \text{ equidistributed}}$$

Examples:

(a) Equidistributed: $\{0, \frac{1}{2}, 0, \frac{1}{3}, \frac{2}{3}, 0, \frac{1}{4}, \dots\}$, $\quad \{\langle n \frac{\sqrt{2}}{2} \rangle\}_{n=1}^{\infty}$

enumeration of $(0,1)\cap\mathbb{Q}$

(b) Not: $\{\langle n\gamma\rangle\}_{n=1}^{\infty} \overset{\gamma = \frac{p}{q}}{=} \underbrace{\{0, \gamma, \dots, \langle q\gamma \rangle\}}_{\text{finite set}}$; $\quad \underbrace{\{\xi_n\}_{n=1}^{\infty}}_{\text{dense}}$, $\xi = \begin{cases} r_{\frac{n}{2}}, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$